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APPLICATION OF SYMMETRY METHOD TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

Fields of industrial and engineering are concerned with the dynamics of fluids. In particular, the study of the dynamical behavior of a fluid near a surface, this is known as boundary layer problems. There are many differential equations arising in boundary layer theory, here we study two of them. One of these differential equations is the boundary layer equation (BLE) of the third order autonomous nonlinear differential equation

$$f''' + \beta f f'' - \alpha f'^2 = 0.$$

The other is the Falkner-Skan equation(FSE) which is also of the third order autonomous nonlinear differential equation

$$f''' + f f'' + 2(1 - f'^2) = 0.$$

In this research we analyze these boundary layer equations using a group theoretical method known as symmetry method. At first, we outline the basic concepts and definitions of the method. Second, we obtain the symmetry groups admitted by the boundary layer equations. Third, we reduce the order of the boundary layer equation to quadrature, also we reduce the order of the Falkner-Skan equation to the second order using the symmetry groups. Furthermore, we give a general solution of the boundary layer equation for the special case when $\beta = 2\alpha$. Finally, we construct invariant solutions of the boundary layer equation from the symmetry group. To our knowledge, there has been no study concerning the symmetry analysis of these equations in the literature.

Dedication

With all my love I dedicate this research to my family who have been my constant source of inspiration. They have given me the motivation and discipline to tackle any task with enthusiasm and determination. Without their love and support this research would not have been made possible.

Acknowledgment

I am grateful to Allah for guiding me to develop and complete this research. Indeed, without his help and will, nothing is accomplished.

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CHAPTER 1

FUNDAMENTAL CONCEPTS OF LIE GROUPS

1.0 Introduction

Lie group of transformations is an important tool since it is used to find the invariant solutions of differential equations. There are methods to solve the ordinary differential equations such as separable, homogeneous or exact equations using special techniques. Lie discovered that these techniques are special cases of a general method based on Lie group. This general method is applicable to linear and nonlinear differential equations, and here lies the importance of this method. Section 1.1 introduces the basic definitions of the one-parameter Lie group of transformation. Section 1.2 deals with an important formula related to the infinitesimal transformation of a one-parameter Lie group. The relation between the global transformation and the infinitesimal transformation and how we may find this relation are presented in Section 1.3. The extended infinitesimal transformation is given in Section 1.4. Sections 1.5 demonstrates the concept of invariance.

1.1 One parameter Lie Group of Transformations

Before defining the one-parameter Lie group of transformations we will define the group in general, which is one of the basic concepts in abstract algebra.

Definition 1.1.1. A set of elements G with binary operation $*$ between its elements is said to be a *group* if the following are satisfied :

(i) If a and b are two elements of G, then $a*b$ is also an element of G.

(ii) The operation $*$ is associative, i.e.

$$a*(b*c) = (a*b)*c.$$

(iii) There exists a unique element e of G such that for any element a of G,

$$a*e = e*a = a,$$

e is called the identity element.

(iv) For any element a of G, there exists an element a^{-1}, such that

$$a*a^{-1} = a^{-1}*a = e,$$

$a^{-1}$the inverse of a.

Examples 1.1.2.

(1) The set of integers Z with the operation $a*b= a+b$ forms a group.

(2) The set of all non-zero complex numbers with the operation $a*b=a.b$ forms a group.

Definition 1.1.3. ([10], Definition 1.2.3). Let H be the set of transformations

$$T_a : x^* = \phi(x,y,a), \quad (1.1.1a)$$

$$y^* = \psi(x,y,a). \quad (1.1.1b)$$

in the (x,y) plane where $a \in$ R. H with the successive transformation as an operation on it is said to be a *one- parameter group of transformations* if the operation on H defines a Law of composition.

$$T_a T_b : x^{**} = \phi(x^*,y^*,b) = \phi(x,y,r(a,b)),$$

$$y^{**} = \psi(x^*,y^*,b) = \psi(x,y,r(a,b)),$$

and if the following axioms hold:

(i) (Closure). If $T_a, T_b \in$ H, then $T_a T_b = T_{r(a,b)} \in$ H.

(ii) (Associative Law). If $T_a, T_b, T_c, \in$ H, then $T_a(T_b T_c) = (T_a, T_b) T_c$.

(iii) (Identity). There exists an identity transformation

$$T_e : x^* = \phi(x,y,a) = x,$$
$$y^* = \psi(x,y,a) = y,$$

which is characterized by the value $a = e$.

(iv) (Inverse). For every $T_a \in H$ there exists an inverse transformation

$$T_{a^{-1}} : x^* = \phi(x,y,a^{-1}),$$
$$y^* = \psi(x,y,a^{-1}),$$

which is characterized by the parameter (a^{-1}).

A group of transformations T_a ;$a \in R$ defines a *one-parameter Lie group of transformations* if in addition to the above axioms (i) – (iv) it satisfies the following conditions:

(v) a is continuous parameter. Without loss of generality $a = 0$ corresponds identity element.

(vi) $\phi(x,y,a)$ and $\psi(x,y,a)$ are infinitely differentiable with respect to x, y and they are analytic functions in $a \in R$.

(vii) $r(a,b)$ is an analytic function of a,b.

Examples 1.1.4.

(1) A group of translations in the plane

$$T_a : x^* = x,$$
$$y^* = y + a,\ a \in R$$

is an example of a one-parameter Lie group of transformations and corresponds to motions parallel to the y- axis.

(2) A set H of transformation

$$T_a : x^* = e^a(x \cos a - y \sin a),$$
$$y^* = e^a (x \sin a + y \cos a),$$

where $a \in R$.

(i) The set is closed since

$$\begin{aligned} T_a T_b : x^{**} &= e^b(x^* \cos b - y^* \sin b) \\ &= e^b\{\cos b[e^a(x \cos a - y \sin a)] - \sin b[e^a(x \sin a + y \cos a)]\} \\ &= e^{a+b}[x \cos(a+b) - y \sin(a+b)], \end{aligned}$$

$$y^{**} = e^{b}(x^{*}\sin b + y^{*}\cos b)$$
$$= e^{a+b}[x\sin(a+b) + y\cos(a+b)].$$

(ii)

$$T_a(T_b T_c): x^{***} = e^{a+(b+c)}\{x\cos[a+(b+c)] - y\sin[a+(b+c)]\},$$
$$y^{***} = e^{a+(b+c)}\{x\sin[a+(b+c)] + y\cos[a+(b+c)]\}.$$
$$(T_a T_b)T_c: x^{***} = e^{(a+b)+c}\{x\cos[(a+b)+c] - y\sin[(a+b)+c]\},$$
$$y^{***} = e^{(a+b)+c}\{x\sin[(a+b)+c] + y\cos[(a+b)+c]\}.$$

Since $a+(b+c) = (a+b)+c$, $T_a(T_b T_c) = (T_a T_b)T_c$. Hence the associative law is satisfied.

(iii) If $a = 0$ we have the identity transformation

$$T_0: x^{*} = x,$$
$$y^{*} = y.$$

(iv)

$$T_a T_{-a}: x^{**} = e^{-a}(x^{*}\cos(-a) - y^{*}\sin(-a))$$
$$= e^{a-a}(x\cos(a-a) - y\sin(a-a))$$
$$= x,$$
$$y^{**} = e^{-a}(x^{*}\sin(-a) + y^{*}\cos(-a))$$
$$= e^{a-a}(x\sin(a-a) + y\cos(a-a))$$
$$= y.$$

Hence the inverse transformation is

$$T_{a^{-1}}: x^{*} = e^{-a}(x\cos(-a) - y\sin(-a)),$$
$$y^{*} = e^{-a}(x\sin(-a) + y\cos(-a)).$$

So, H with the successive transformation forms a one-parameter Lie group of transformations since the definition 1.1.3 is satisfied. This group is called a *rotation group.*

Example 1.1.5. Let H be the set of transformations

$$T_a : x^* = x + \sqrt{a},$$
$$y^* = y.$$

H is not closed since

$$T_a T_b : x^{**} = x^* + \sqrt{b} = x + \sqrt{a} + \sqrt{b},$$

and $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Hence, H does not form a one-parameter Lie group of transformations.

1.2 Infinitesimal Transformations

In this section, we will deduce infinitesimal transformation from a one-parameter Lie group of transformation. Moreover, we will define an important formula called the infinitesimal generator which can be used to identify the infinitesimal transformation and vice-versa.

In group theoretical methods, a one-parameter Lie group of transformations [7] may take the form as global or infinitesimal transformations. A *global transformation* has the form

$$x^* = \phi(x,y,a), \tag{1.2.1a}$$
$$y^* = \psi(x,y,a). \tag{1.2.1b}$$

The infinitesimal form is derived by expanding (1.2.1a,b) in Taylor series about a=0 where 0 is the value of the parameter corresponding to the identity transformation

$$x^* = \phi(x,y,0) + (a-0)\left(\frac{\partial \phi}{\partial a}\right)_{a=0} + \frac{(a-0)^2}{2!}\left(\frac{\partial^2 \phi}{\partial a^2}\right)_{a=0} + \dots,$$
$$y^* = \psi(x,y,0) + (a-0)\left(\frac{\partial \psi}{\partial a}\right)_{a=0} + \frac{(a-0)^2}{2!}\left(\frac{\partial^2 \psi}{\partial a^2}\right)_{a=0} + \dots.$$

Letting $\varepsilon = a - 0$ where ε is an infinitesimal quantity, we have

$$x^* = x + \varepsilon\xi(x,y) + o(\varepsilon^2), \tag{1.2.2a}$$
$$y^* = y + \varepsilon\eta(x,y) + o(\varepsilon^2), \tag{1.2.2b}$$

where $\xi\ (x,y) = (\frac{\partial \phi}{\partial a})_{a=0}$ and $\eta\ (x,y) = (\frac{\partial \psi}{\partial a})_{a=0}$. The transformation (1.2.2a,b) is called the *infinitesimal transformation* and (1.2.1a,b) is called the *global transformation* of the group.

Example 1.2.1. Consider the one-parameter Lie group of transformations

$$T_a:\ x^* = ax,$$
$$y^* = (y+a).$$

To obtain the infinitesimal transformation, we first evaluate the infinitesimals:

$$\xi\ (x,y) = (\frac{\partial x^*}{\partial a})_{a=0} = x,$$
$$\eta\ (x,y) = (\frac{\partial y^*}{\partial a})_{a=0} = 1.$$

Hence the infinitesimal transformations is

$$x^* = x + \varepsilon x + o(\varepsilon^2),$$
$$y^* = y + \varepsilon + o(\varepsilon^2).$$

Definition 1.2.2. The infinitesimal generator of the one-parameter Lie group of transformations is the operator.

$$X = \xi\ (x,y)\frac{\partial}{\partial x} + \eta\ (x,y)\frac{\partial}{\partial y}.$$

Note that, for any differentiable function $F(x,y)$:

(i) $XF(x,y) = \xi\ (x,y)\ \frac{\partial F(x,y)}{\partial x} + \eta\ (x,y)\ \frac{\partial F(x,y)}{\partial y}$.

(ii) $Xx = \xi\ (x,y)\frac{\partial (x)}{\partial x} + \eta\ (x,y)\frac{\partial (x)}{\partial y} = \xi\ (x,y)$.

(iii) $X^k = XX^{k-1}$.

(iv) $X^0F(x,y) = F(x,y)$.

In Example 1.2.1, we found the infinitesimal transformations. The infinitesimal generator of this transformations is:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

1.3 Relation Between The Global Transformations and The Infinitesimal Transformations

It can be seen from Section 1.2 that there is a relation between the infinitesimal transformations and the global transformations. In this section we will introduce theorems that determine specifically the relation between the two transformations.

Let $\mathbf{x} = (x,y)$ lie in region $D \subset R^2$ and suppose that the one-parameter Lie group of transformations is defined by

$$\mathbf{x}^* = \phi(\mathbf{x},a),$$

with the infinitesimals $\xi(\mathbf{x})$. Before stating the first fundamental theorem of Lie, we will start by the following useful lemma.

Lemma 1.3.1.([8], Lemma 2.2.1-1)

$$\phi(\mathbf{x},a + \Delta a) = \phi(\phi(\mathbf{x},a),\ r(a^{-1},a + \Delta a))$$

Proof: We start the proof from the right hand side

$$\phi(\phi(\mathbf{x},a),\ r(a^{-1},a + \Delta a)) \ = \phi(\mathbf{x}^*,\ r(a^{-1},a + \Delta a)),$$

but

$$\phi(\mathbf{x}^*,b) \ = \phi(\mathbf{x},r(a,b)). \tag{1.3.1}$$

Using (1.3.1), we obtain that:

$$\phi(\mathbf{x}^*,\ r(a^{-1},a + \Delta a)) \ = \phi\ (\mathbf{x},r(a,\ r(a^{-1},a + \Delta a))).$$

Since r is associative, the above equation becomes

$$\begin{aligned}\phi(\mathbf{x}^*,\ r(a^{-1},a + \Delta a)) \ &= \phi(\mathbf{x},r(r(a,a^{-1}),a + \Delta a) \\ &=\phi\ (\mathbf{x},r(0,a + \Delta a)).\end{aligned}$$

where $r(a^{-1},a) = 0$ is the identity of r.

Hence,

$$\phi(\phi(\mathbf{x},a),\ r(a^{-1},a + \Delta a)) = \phi(\mathbf{x},a + \Delta a).$$

Theorem 1.3.2.(First Fundamental Theorem of Lie, [8]). There exists a parameterization $\tau(a)$ such that the Lie group of transformations $\mathbf{x}^* = \phi(\mathbf{x},a)$ is equivalent to the solution of the initial value problems for the system of first order ordinary differential equation.

$$\frac{d\mathbf{x}^*}{d\tau} = \xi\,(\mathbf{x}^*),$$

with

$$\mathbf{x}^* = \mathbf{x} \text{ when } \tau = 0.$$

In particular

$$\tau\,(a) = \int_0^a \Gamma\,(a`)\,da`,$$

where

$$\Gamma(a) = \left(\frac{\partial r\,(\varepsilon,\delta)}{\partial \delta}\right)_{(\varepsilon,\delta)=(a^{-1},a)}.$$

Proof: Suppose that there is an infinitesimal change Δa in the transformations $\phi(\mathbf{x},a+\Delta a)$. Expanding the transformation in power series about $\Delta a = 0$

$$\phi(\mathbf{x},a+\Delta a) = \phi(\mathbf{x},a) + \left(\frac{\partial \phi(\mathbf{x},a+\Delta a)}{\partial (a+\Delta a)}\right)_{\Delta a=0} (\Delta a - 0) + o((\Delta a)^2)$$

$$= \mathbf{x}^* + \frac{\partial \phi(\mathbf{x},a)}{\partial a}\Delta a + o((\Delta a)^2. \tag{1.3.2a}$$

From lemma 1.3.1, we have

$$\phi(\mathbf{x},a+\Delta a) = \phi(\phi(\mathbf{x},a), r(a^{-1},a+\Delta a)).$$

First we will expand $r(a^{-1}, a+\Delta a)$ in a power series about $\Delta a = 0$

$$r(a^{-1},a+\Delta a) = r(a^{-1},a) + \left(\frac{\partial r\,(a^{-1},a+\Delta a)}{\partial (a+\Delta a)}\right)_{\Delta a=0} (\Delta a - 0) + o((\Delta a)^2)$$

$$= 0 + \frac{\partial r\,(a^{-1},a)}{\partial a}\Delta a + o((\Delta a)^2)$$

$$= \Gamma(a)\,\Delta a + o((\Delta a)^2),$$

Hence rewriting lemma 1.3.1 as:

$$\phi(\mathbf{x},a+\Delta a) = \phi(\phi(\mathbf{x},a), r(a^{-1},a+\Delta a))$$

$$= \phi(\phi(\mathbf{x},a), \Gamma(a)\,\Delta a + o((\Delta a)^2)$$

$$= \phi(\mathbf{x}^*, \Gamma(a)\,\Delta a + o((\Delta a)^2).$$

Letting, $b = \Gamma(a)\,\Delta a$, we have $b = 0$ when $\Delta a = 0$. Expanding the right hand side of the above equation in power series about $b = 0$ ($\Delta a = 0$), we obtain:

$$\phi(\mathbf{x},a+\Delta a) = (\phi(\mathbf{x}^*, \Gamma(a)\,\Delta a))_{b=0} + \left(\frac{\partial\phi(\mathbf{x}^*,b)}{\partial b}\right)_{b=0} b + o((\Delta a)^2)$$

$$= \phi(\mathbf{x}^*,0) + \Gamma(a)\,\Delta a\,\xi\,(\mathbf{x}^*) + o((\Delta a)^2) \qquad (1.3.2b)$$

Equating (1.3.2a) and (1.3.2b), we get:

$$\frac{\partial\phi(\mathbf{x},a)}{\partial a} = \Gamma(a)\,\xi\,(\mathbf{x}^*),$$

$$\frac{d\mathbf{x}^*}{da} = \Gamma(a)\,\xi\,(\mathbf{x}^*),$$

with

$$\mathbf{x}^* = \mathbf{x} \text{ at } a = 0.$$

In particular

$$\tau\,(a) = \int_0^a \Gamma\,(a`)\,da` \Rightarrow \frac{d\tau}{da} = \Gamma(a).$$

Hence,

$$\frac{d\mathbf{x}^*}{da} = \frac{d\tau}{da}\xi\,(\mathbf{x}^*),$$

$$\frac{d\mathbf{x}^*}{d\tau} = \xi\,(\mathbf{x}^*),$$

with

$$\mathbf{x}^* = \mathbf{x} \text{ at } a = 0.$$

From the first fundamental theorem of Lie we conclude that the infinitesimal transformation contains the essential information determining a one-parameter Lie group of transformations.

Example 1.3.3. For the group of transformations

$$x^* = x + a, \qquad (1.3.3a)$$

$$y^* = y, \qquad (1.3.3b)$$

the law of composition is $r(\varepsilon,\delta) = \varepsilon + \delta$, and $a^{-1} = -a$. Then $\frac{\partial r\,(\varepsilon,\delta)}{\partial\delta} = 1$, and $\Gamma(a) = 1$.

Let $\mathbf{x} = (x,y)$, then the group (1.3.3a,b) is $\phi(\mathbf{x},a) = (x + a, y)$. Thus $\frac{\partial\phi(\mathbf{x},a)}{\partial a} = (1,0)$.

Hence,

$$\xi(\mathbf{x}) = \left(\frac{\partial \phi(\mathbf{x},a)}{\partial a}\right)_{a=0} = (1,0).$$

By using the first fundamental theorem of Lie, we have:

$$\frac{dx^*}{da} = 1, \quad \frac{dy^*}{da} = 0,$$

with $x^* = x$, $y^* = y$ at $a = 0$. The solution of this initial value problem is equivalent to the group of transformations (1.3.3a,b).

Theorem 1.3.4. ([8], Theoremn2.2.3-1). The one-parameter Lie group of transformations (1.2.1a,b) is equivalent to

$$\mathbf{x}^* = e^{a\mathrm{X}}\mathbf{x} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\mathrm{X}^k\mathbf{x},$$

where the operator $\mathrm{X} = \sum_{i=1}^{2} \xi_i(\mathbf{x}) \frac{\partial}{\partial x_i}$ and $\mathrm{X}^k = \mathrm{X}\mathrm{X}^{k-1}$.

Proof: Let

$$\mathrm{X} = \mathrm{X}(\mathbf{x}) = \sum_{i=1}^{2} \xi_i(\mathbf{x}) \frac{\partial}{\partial x_i},$$

$$\mathrm{X}(\mathbf{x}^*) = \sum_{i=1}^{2} \xi_i(\mathbf{x}^*) \frac{\partial}{\partial x_i},$$

where $\mathbf{x}^* = \phi(\mathbf{x},a)$. Expanding $\mathbf{x}^* = \phi(\mathbf{x},a)$ in Taylor series about $a = 0$,

$$\mathbf{x}^* = \sum_{k=0}^{\infty} \frac{a^k}{k!}\left(\frac{\partial^k \phi(\mathbf{x},a)}{\partial a^k}\right)_{a=0} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\left(\frac{\partial^k \mathbf{x}^*}{\partial a^k}\right)_{a=0}. \qquad (1.3.4)$$

For any function $\mathrm{F}(\mathbf{x})$:

$$\frac{d\mathrm{F}(\mathbf{x}^*)}{da} = \sum_{i=1}^{2} \frac{\partial \mathrm{F}(\mathbf{x}^*)}{\partial {x_i}^*}\frac{d{x_i}^*}{da}$$

$$= \sum_{i=1}^{2} \xi_i(\mathbf{x}^*) \frac{\partial \mathrm{F}(\mathbf{x}^*)}{\partial {x_i}^*}$$

$$= \mathrm{X}(\mathbf{x}^*)\,\mathrm{F}(\mathbf{x}^*).$$

Hence, it follows that:

$$\frac{d\mathbf{x}^*}{da} = \mathrm{X}(\mathbf{x}^*)\,\mathbf{x}^*,$$

$$\frac{d^2\mathbf{x}^*}{da^2} = \frac{d}{da}\left(\frac{d\mathbf{x}^*}{da}\right) = \mathrm{X}(\mathbf{x}^*)\,\mathrm{X}(\mathbf{x}^*)\,\mathbf{x}^* = \mathrm{X}^2(\mathbf{x}^*)\,\mathbf{x}^*.$$

Thus,

$$\frac{d^k \mathbf{x}^*}{da^k} = \mathrm{X}^k(\mathbf{x}^*)\,\mathbf{x}^*.$$

Substituting the above equation into (1.3.4), we have:

$$\mathbf{x}^* = \sum_{k=0}^{\infty} \frac{a^k}{k!} \mathrm{X}^k \mathbf{x},$$

where $\left(\frac{d^k \mathbf{x}^*}{da^k}\right)_{a=0} = (\mathrm{X}^k \mathbf{x}^*)_{a=0} = \mathrm{X}^k \mathbf{x}$.

From section 1.2 and the above theorems the relation between the infinitesimal transformations and the global transformations can be deduced by one of the following:

(1) If a global transformations (1.1,1a,b) is given, we can obtain the infinitesimal transformations by evaluating

$$\xi\,(x,y) = \left(\frac{\partial \phi}{\partial a}\right)_{a=0} \text{ and } \eta\,(x,y) = \left(\frac{\partial \psi}{\partial a}\right)_{a=0},$$

and then substituting into (1.2.1a,b).

(2) If an infinitesimal transformations (1.2.1a,b) is given, then there are two ways to obtain the global transformations:

(i) Solve the initial value problem

$$\frac{dx^*}{da} = \xi\,(x^*,y^*),$$

$$\frac{dy^*}{da} = \eta(x^*,y^*),$$

with $x^* = x$, $y^* = y$, at $a = 0$.

(ii) Express the global transformations group in terms of a power series

$$x^* = \sum_{k=0}^{\infty} \frac{a^k}{k!} \mathrm{X}^k x,$$

$$y^* = \sum_{k=0}^{\infty} \frac{a^k}{k!} \mathrm{X}^k y,$$

where X is the infinitesimal generator.

Example 1.3.5. Consider the infinitesimal transformation

$$x^* = x + \varepsilon y + o(\varepsilon^2),$$

$$y^* = y - \varepsilon x + o(\varepsilon^2),$$

with infinitesimal generator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Then

$$\xi(x,y) = y,$$

$$\eta(x,y) = -x.$$

To obtain the global transformations group, we solve the initial value problem

$$\frac{dx^*}{da} = y^*, \quad (1.3.5.a)$$

$$\frac{dy^*}{da} = -x^*. \quad (1.3.5.b)$$

Differentiating (1.3.5a) and substituting into (1.3.5b), we have:

$$\frac{d^2x^*}{da^2} = \frac{dy^*}{da} = -x^*,$$

$$\frac{d^2x^*}{da^2} + x^* = 0,$$

which gives the solution $x^* = A\cos a + B\sin a$, $y^* = -A\sin a + B\cos a$. By satisfying the initial conditions $x^* = x, y^* = y$ at $a = 0$, we obtain that:

$$A = x, \text{ and } B = y.$$

Hence, the global transformation is:

$$x^* = x\cos a + y\sin a,$$

$$y^* = y\cos a - x\sin a.$$

Example1.3.6. Consider the infinitesimal generator

$$X = x\frac{\partial}{\partial x},$$

we obtain the global transformations group by using Theorem 1.3.4.

$$x^* = \sum_{k=0}^{\infty}\frac{a^k}{k!}X^k x$$

$$= x + a(x\frac{\partial}{\partial x})\,x + \frac{a^2}{2!}(x\frac{\partial}{\partial x})(x\frac{\partial}{\partial x})\,x + \ldots$$

$$= x + a\,x + \frac{a^2}{2!}x + \ldots$$

$$= [1 + a + \frac{a^2}{2!} + \ldots]\,x$$

$$= e^{a} x,$$

$$y^* = \sum_{k=0}^{\infty} \frac{a^k}{k!} \mathrm{X}^{\mathrm{k}} y$$

$$= y + a(\mathrm{x}\frac{\partial}{\partial x}) y + \ldots = y.$$

Hence, the global transformation is:

$$x^* = e^{a} x,$$

$$y^* = y.$$

1.4 Extended Infinitesimal Transformations

The extended infinitesimal transformations is important in determining the symmetry group of a differential equation which will be seen in chapter 2. We will consider the extended infinitesimal transformations for the case of one dependent variable and one independent variable.

Definition 1.4.1. ([8], Definition 2.3.1-1). The total derivative operator is defined by

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \ldots + y_{n+1} \frac{\partial}{\partial y_n} + \ldots, \tag{1.4.1}$$

where $y_i = \frac{dy_{i-1}}{dx}$, and $y_0 = y$.

Theorem 1.4.2. ([8], Theorem 2.3.1-4). The kth extension of the one parameter Lie group of transformations

$$x^* = \phi(x, y, \varepsilon),$$

$$y^* = \psi(x, y, \varepsilon),$$

is given by:

$$x^* = \phi(x, y, \varepsilon),$$

$$y^* = \psi(x, y, \varepsilon),$$

$$y_i^* = \psi_i(x, y, y_1, \ldots, y_i, \varepsilon) = \frac{D\psi_{i-1}}{Dx} \div \frac{D\phi(x, y, \varepsilon)}{Dx},\ i = 1,2,\ldots,k \tag{1.4.2}$$

where $\psi_0 = \psi(x,y,\varepsilon)$.

Applying the above theorem in obtaining the first extended infinitesimal transformation for

$$x^* = \phi(x, y, \varepsilon), \quad (1.4.3a)$$

$$y^* = \psi(x, y, \varepsilon), \quad (1.4.3b)$$

where y is the dependent variable and x is the independent variable. The differential coefficient $y_1 = \frac{dy}{dx}$ can be consider as a third variable, then we have

$$y_1^* = \frac{dy^*}{dx^*},$$

where

$$dy^* = d\psi\,(x, y, \varepsilon) = \frac{\partial\,\psi(x,y,\varepsilon)}{\partial x}\,\mathrm{d}x + \frac{\partial\,\psi\,(x,y,\varepsilon)}{\partial y}\,\mathrm{d}y,$$

$$dx^* = d\phi(x, y, \varepsilon) = \frac{\partial\,\phi(x,y,\varepsilon)}{\partial x}\,\mathrm{d}x + \frac{\partial\,\phi(x,y,\varepsilon)}{\partial y}\,\mathrm{d}y,$$

then

$$\begin{aligned} \mathrm{d}y^* &= (0 + \varepsilon\eta_x + o(\varepsilon^2))\,\mathrm{d}x + (1 + \varepsilon\eta_y + o(\varepsilon^2))\,\mathrm{d}y \\ &= dy + \varepsilon(\eta_x dx + \eta_y\, dy) + o(\varepsilon^2), \\ \mathrm{d}x^* &= (1 + \varepsilon\xi_x + o(\varepsilon^2))\,\mathrm{d}x + (0 + \varepsilon\xi_y + o(\varepsilon^2))\,\mathrm{d}y \\ &= \mathrm{d}x + \varepsilon(\xi_x\,\mathrm{d}x + \xi_y\,\mathrm{d}y) + o(\varepsilon^2). \end{aligned}$$

Hence,

$$\begin{aligned} y_1^* &= \frac{dy + (\varepsilon\eta_x dx + \varepsilon\eta_y\, dy) + o(\varepsilon^2)}{dx + (\varepsilon\xi_x dx + \varepsilon\xi_y\, dy) + o(\varepsilon^2)} \\ &= y_1 + \varepsilon[\eta_x + (\eta_y - \xi_x)y_1 - \xi_y y_1^2] + o(\varepsilon^2). \end{aligned}$$

Now

$$x^* = x + \varepsilon\xi\,(x,y) + o(\varepsilon^2),$$

$$y^* = y + \varepsilon\eta\,(x,y) + o(\varepsilon^2),$$

$$y_1^* = y_1 + \varepsilon\eta^{(1)}(x,y,y_1) + o(\varepsilon^2),$$

form a group called the *first extended infinitesimal transformations* with infinitesimals

$$(\xi(x,y), \eta(x,y), \eta^{(1)}(x,y,y_1))$$

and corresponding first extended infinitesimal generator

$$X^{(1)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1},$$

where $\eta^{(1)}(x,y,y_1) = \eta_x + (\eta_y - \xi_x)\, y_1 - \xi_y y_1^2$.

The kth extension of (1.4.3a,b), is given by

$$x^* = \phi(x,y,\varepsilon) = x + \varepsilon\xi(x,y) + o(\varepsilon^2), \quad (1.4.4a)$$

$$y^* = \psi(x,y,\varepsilon) = y + \varepsilon\eta(x,y) + o(\varepsilon^2), \quad (1.4.4b)$$

$$y_1^* = \psi_1(x,y,y_1,\varepsilon) = y_1 + \varepsilon\eta^{(1)}(x,y,y_1) + o(\varepsilon^2), \quad (1.4.4c)$$

$$\vdots \qquad\qquad \vdots$$

$$y_k^* = \psi_k(x,y,y_1,\ldots,y_k,\varepsilon) = y_k + \varepsilon\eta^{(k)}(x,y,y_1,\ldots,y_k) + o(\varepsilon^2), \quad (1.4.4d)$$

with (kth extended) infinitesimals

$$(\xi(x,y), \eta(x,y), \eta^{(1)}(x,y,y_1), \ldots, \eta^{(k)}(x,y,y_1,\ldots,y_k)),$$

and corresponding (kth extended) infinitesimal generator

$$X^{(k)} =$$

$$\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}((x,y,y_1)\frac{\partial}{\partial y_1} + \ldots + \eta^{(k)}(x,y,y_1,\ldots,y_k)\frac{\partial}{\partial y_k},$$

where k= 1,2,…

The following theorem is useful to obtain explicit formulas for the extended infinitesimal $\{\eta^{(k)}\}$.

Theorem 1.4.3. ([8], Theorem 2.3.2-1).

$$\eta^{(k)}(x,y,y_1,\ldots,y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}, \ k=1,2,\ldots,$$

where

$$\eta^{(0)} = \eta(x,y).$$

Proof: From (1.4.2.), (1.4.4a-d),and (1.4.1), we have:

$$\psi_k(x,y,y_1,\ldots,y_k,\varepsilon) = \frac{D\psi_{k-1}}{Dx} \div \frac{D\phi(x,y,\varepsilon)}{Dx}$$

$$= \frac{D[y_{k-1}+\varepsilon\eta^{(k-1)}+o(\varepsilon^2)]}{Dx} \div \frac{D[x+\varepsilon\xi(x,y)+o(\varepsilon^2)]}{Dx}$$

$$=[y_k + \varepsilon\frac{D\eta^{k-1}}{Dx}] \div [1 + \varepsilon\frac{D\xi(x,y)}{Dx}]$$

$$=y_k + \varepsilon[\frac{D\eta^{k-1}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}] + o(\varepsilon^2)$$

$$= y_k + \varepsilon\eta^{(k)} + o(\varepsilon^2).$$

Consider the formulas of the extended infinitesimal $\eta^{(1)}$,$\eta^{(2)}$ and $\eta^{(3)}$.

$$\eta^{(1)} = \eta_x + (\eta_x - \xi_x)\, y_1 - \xi_y y_1^2,$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_1 + (\eta_{yy} - 2\xi_{xy})(y_1)^2 - \xi_{yy}(y_1)^3 + (\eta_y - 2\xi_x)y_2 - 3\xi_y y_1 y_2.$$

$$\eta^{(3)} = \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_1 + 3(\eta_{xxy} - \xi_{xxy})(y_1)^2 + (\eta_{yyy} - 3\xi_{xyy})(y_1)^3 - \xi_{yyy}(y_1)^4 + 3(\eta_{xy} - \xi_{xx})y_2 + 3(\eta_{yy} - 3\xi_{xy})y_1 y_2 - 6\xi_{yy}(y_1)^2 y_2 - 3\xi_y(y_2)^2 + (\eta_y - 3\xi_x)y_3 - 4\xi_y y_1 y_3.$$

1.5 The Concept Of Invariance

In this section, a definition of invariance is presented which is useful for reducing the order of the ordinary differential equations to obtain a solution.

Definition 1.5.1. ([10],Definition 1.5.1). A function $g(x,y)$ is said to be *invariant* under the one-parameter Lie group of transformations (1.1.1a,b), if it is unaltered by the transformations (1.1.1a,b), that is

$$g(x^*,y^*) = g(x,y).$$

Theorem 1.5.2. ([8], Theorem 2.2.4-1). A function $g(x,y)$ is invariant under (1.1.1a,b)if and only if

$$\mathrm{X}g = \xi(x,y)\frac{\partial g}{\partial x} + \eta(x,y)\frac{\partial g}{\partial y} = 0 \qquad (1.5.1)$$

The invariant function $g(x,y)$ under the one-parameter Lie group of transformations can be found by solving the differential equation (1.5.1).

Example 1.5.3. Consider the infinitesimal generator

$$\mathrm{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The invariant function $g(x,y)$ can be obtained by solving the partial differential equation

$$-y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = 0.$$

The auxiliary equation is

$$\frac{dx}{-y} = \frac{dy}{x}.$$

Therefore

$$g(x,y) = x^2 + y^2 = c,$$

where c is an arbitrary constant.

We will demonstrate how invariance under the one-parameter Lie group of transformations can be used to reduce the order of the ordinary differential equation to obtain a solution.

Consider the differential equation

$$\frac{dy}{dx} = F(x,y) \tag{1.5.2}$$

under this special transformation

$$T_a : x^* = ax, \tag{1.5.3a}$$

$$y^* = a^k y. \tag{1.5.3b}$$

Assume that the differential equation is invariant under the transformation (1.5.3a,b). The differential equation (1.5.2) is said to be invariant under the transformation (1.5.3a,b) if its read the same in the new coordinates i.e.:

$$\frac{dy^*}{dx^*} = F(x^*,y^*).$$

Now,

$$x = \frac{x^*}{a},\ y = \frac{y^*}{a^k}.$$

$$dx = \frac{dx^*}{a},\ dy = \frac{dy^*}{a^k}.$$

substitute into (1.5.2)

$$\frac{1}{a^{k-1}}\frac{dy^*}{dx^*} = F\left(\frac{x^*}{a}, \frac{y^*}{a^k}\right),$$

$$\frac{dy^*}{dx^*} = a^{k-1}F\left(\frac{x^*}{a}, \frac{y^*}{a^k}\right).$$

For equation (1.5.2) to be invariant under group of transformation (1.5.3a,b) we must have

$$F(x^*, y^*) = a^{k-1}F\left(\frac{x^*}{a}, \frac{y^*}{a^k}\right),$$

where in terms of x and y, we obtain

$$F(ax, a^k y) = a^{k-1} F(x,y),$$

$$F(x,y) = \frac{1}{a^{k-1}} F(ax, a^k y) \qquad (1.5.4)$$

Differentiating equation (1.5.4) with respect to a:

$$0 = (1-k)\, a^{-k}F(ax, a^k y) + a^{1-k}\left(\frac{\partial F(x^*,y^*)}{\partial (ax)} x + \frac{\partial F(x^*,y^*)}{\partial a^k y} k a^{k-1} y \right),$$

$$(1-k)a^{-k}F(ax, a^k y) + a^{1-k} x \frac{\partial F(x^*,y^*)}{\partial x^*} + k\, y \frac{\partial F(x^*,y^*)}{\partial y^*} = 0,$$

$$(1-k)\, F(x^*, y^*) + x^* \frac{\partial F(x^*,y^*)}{\partial x^*} + k y^* \frac{\partial F(x^*,y^*)}{\partial y^*} = 0,$$

$$x^* \frac{\partial F(x^*,y^*)}{\partial x^*} + k\, y^* \frac{\partial F(x^*,y^*)}{\partial y^*} = (k-1)\, F(x^*, y^*). \qquad (1.5.5)$$

The auxiliary equation is:

$$\frac{dx^*}{x^*} = \frac{dy^*}{ky^*} = \frac{dF}{(k-1)F(x^*,y^*)}.$$

Integration of the first two equations gives:

$$\ln x^* + \ln c = \frac{1}{k} \ln y^*,$$

$$k(\ln x^* + \ln c) = \ln y^*$$

$$y^* = (x^*)^k c.$$

Hence,

$$c = \frac{y^*}{x^{*k}} .$$

The integration of the first and third equations gives the general solution of (1.5.5) :

$$(k-1) \frac{dx^*}{x^*} = \frac{dF(x^*,y^*)}{F(x^*,y^*)},$$

$$(k-1)[\ln x^* + \ln g(c)] = \ln F(x^*, y^*),$$

$$F(x^*, y^*) = (x^*)^{k-1} g(c),$$

$$F(x^*, y^*) = (x^*)^{k-1} g\left(\frac{y^*}{x^{*k}}\right).$$

Hence, from invariance we have:

$$F(x, y) = (x)^{k-1}\, g\left(\frac{y}{x^k}\right),$$

The differential equation (1.5.2) becomes:

$$\frac{dy}{dx} = (x)^{k-1}\, g\left(\frac{y}{x^k}\right). \tag{1.5.6}$$

Let $\sigma(x, y) = \frac{y}{x^k}$, which is known as the similarity coordinates

$$y = \sigma x^k \tag{1.5.7}$$

Differentiating (1.5.7) with respect to x:

$$\frac{dy}{dx} = k x^{k-1} \sigma + x^k \frac{\partial \sigma}{\partial x} \tag{1.5.8}$$

Now, equating (1.5.6) and (1.5.8), we have:

$$kx^{k-1}\sigma + {}^k x \frac{\partial \sigma}{\partial x} = x^{k-1} g(\sigma),$$

$$k\,\sigma + x\frac{d\sigma}{dx} = g(\sigma),$$

$$x\frac{d\sigma}{dx} = g(\sigma) - k\,\sigma,$$

$$x d\sigma = (g(\sigma) - k\,\sigma)dx,$$

$$\frac{d\sigma}{g(\sigma) - k\,\sigma} = \frac{dx}{x}. \tag{1.5.9}$$

By solving (1.5.9) we find σ in terms of x, then substituting back into (1.5.7) we obtain the solution of differential equation (1.5.2) when F(x, y) is of the form given by (1.5.6).

CHAPTER 2

INVARIANCE OF ORDINARY DIFFERENTIAL EQUATIONS

2.0 Introduction

Similarity method is a group theoretical method for simplifying differential equations. This method depends on Lie group of transformations. The object of this Chapter is to present the application of Lie group on differential equation (similarity methods). Section 2.1 demonstrates a method for determining similarity transformation (symmetry group). Section 2.2 presents how invariance of a differential equation under a Lie group of transformation reduces its order. The methods for constructing special solutions (invariant solutions) to a differential equation are given in Section 2.3.

2.1 Symmetry Groups

The importance of symmetry groups lies in reducing the order of a differential equation and obtaining invariant solutions.

Definition 2.1.1. ([10], Definition 1.7.1). A *symmetry group* is a Lie group of transformations which leaves the differential equation invariant, that is, a group admitted by the differential equation.

Definition 2.1.2. ([10], Definition 1.7.2). An ordinary differential equation

$$G(x, y, y_1, \ldots.. y_k) = 0, \tag{2.1.1}$$

where $y_n = \frac{d^n y}{dx^n}$, n = 1,2,…,k, is said to be *invariant* under a one parameter Lie group of transformations

$$x^* = \phi(x,y,a) \quad = x + \varepsilon\xi(x,y) + o(\varepsilon^2), \tag{2.1.2a}$$

$$y^* = \psi(x,y,a) \quad = y + \varepsilon\eta(x,y) + o(\varepsilon^2), \tag{2.1.2b}$$

if it is unaltered by the transformations, that is

$$G(x^*,y^*,y_1^*,\ldots,y_k^*) = G(x,y,y_1,\ldots,y_k).$$

Theorem 2.1.3. (Infinitesimal Criterion for Invariance of an ODE, [8]). Let

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.1.3}$$

be the infinitesimal generator of (2.1.2a,b). Let

$$X^{(k)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \ldots + \eta^{(k)}(x,y,y_1,\ldots,y_k)\frac{\partial}{\partial y_k}, \tag{2.1.4}$$

be the kth extended infinitesimal generator of (2.1.3) where $\eta^{(n)}(x,y,\ldots,y_n)$, $n = 1,2,\ldots k$, are given by theorem 1.4.3 in terms of $(\xi(x,y),\eta(x,y))$. Then (2.1.2a,b) are admitted by (2.1.1) if and only if

$$X^{(k)}G(x,y,y_1,\ldots,y_k) = 0, \text{ when } G(x,y,y_1,\ldots,y_k) = 0.$$

Utilizing the infinitesimal criterion for invariance on an ODE, leads to a differential equation where the unknowns are the infinitesimals $(\xi(x,y),\eta(x,y))$. Equating the coefficients of $(y_1,\ldots,y_k)$ we obtain several equations known as the *determining equations*. By solving these equations the infinitesimals are identified. Hence the symmetry group of the ODE is obtained.

2.2 Reduction Under Group Invariance

Symmetry method depends on finding the one-parameter Lie group of transformations that is admitted by the differential equation, i.e., that leaves the differential equation invariant [9]. For a first order ordinary differential equation, invariance implies a reduction to quadrature. For higher order ordinary differential equations, invariance implies a reduction in order by one. By using the differential invariants, the order of the differential equation is reduced. Moreover, the invariance of the differential equation, under an r-parameter Lie group of transformations, reduces its order by r. This is presented by Olver in [11]. Hence, invariance under a two-parameter reduced the order by two. Bluman and Kumei [8] presented a method for this reduction.

2.2.1 Reduction of the order of an ODE under a one-parameter group

Consider the kth order ordinary *differential* equation

$$G(x,y,y_1,\ldots,y_k) = 0, \tag{2.2.1}$$

where $y_n = \frac{d^n y}{dx^n}$, n = 1,2,…,k, and let

$$x^* = \phi(x,y,a), \tag{2.2.2a}$$

$$y^* = \psi(x,y,a), \tag{2.2.2b}$$

be a one-parameter Lie group of transformations admitted by (2.2.1) with infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{2.2.3}$$

From section 1.5 it is found that the invariant u(x,y) under (2.2.2a,b) can be obtained by solving Xu = 0. Moreover the kth order differential invariants $v^{(k-1)}(x,y,y_1,\ldots,y_k)$ are determined by solving $X^{(k)}v^{(k-1)} = 0$ where k = 1,2,…,n, $X^{(k)}$ is the kth extension of X and $v^{(0)} = v$.

Theorem 2.2.1. ([10], Theorem 1.5). Let the invariant u(x,y) and the first order differential invariant $v(x,y,y_1)$ be known for a given one-parameter Lie group of transformations. Then the derivative

$$\frac{dv}{du} = \frac{v_x + v_y y_1 + v_{y_1} y_2}{u_x + u_y y_1} = \frac{Dv}{Du},$$

is a second order differential invariant.

Proposition 2.2.2. ([11], Proposition 2.56). Any nth order ordinary differential equation having a one-parameter Lie group of transformations (2.2.2a,b) as a symmetry group is equivalent to an (n-1)st order equation

$$Q(u, \nu, \nu^{(1)}, \ldots, \nu^{(n-1)}) = 0,$$

involving the invariants $u(x,y)$, $\nu(x,y,y_1)$ of the first extension $X^{(1)}$corresponding to the group and their derivatives, where $\nu^{(k-1)} = \dfrac{d\nu^{(k-2)}}{du}$,

$k = 2,\ldots,n$.

The above proposition outlines the method of reduction in order of an ordinary differential equation by using the differential invariants. Re-writing the ordinary differential equation in terms of the differential invariants, the result is an ordinary differential equation which is less in order than the original equation by one. To obtain the solution of the original differential equation, we integrate the solution of the reduced differential equation.

2.2.2 Reduction of the order of an ODE under a two-parameter group

Theorem 2.2.3. ([11], Theorem 2.6.1). Let $G(x,y, y_1,\ldots,y_k) = 0$ be a kth order ordinary differential equation invariant under a two-parameter symmetry group. Then there is a (k – 2)nd order equation

$$K(z, w, w_1, \ldots, w_{n-2}) = 0.$$

The general solution to G can be found by a pair of quadratures from the general solution to K.

The above theorem describes the possibility of the reduction in order directly by two if the ordinary differential equation is invariant under a two-parameter Lie group of transformation. Bluman and Kumei [8] gave us the algorithm of this reduction.

Definition 2.2.4. ([10], Definition 1.8.5). The *commutator* of X_1 and X_2 denoted by $[X_1, X_2]$ is defined as

$$[X_1, X_2] = X_1X_2 - X_2X_1.$$

Theorem 2.2.5. ([8], Theorem 2.4.2-3). Let $X_1^{(k)}$, $X_2^{(k)}$ be the kth extended infinitesimal generators of the infinitesimal generators X_1, X_2 and let

$[X_1,X_2]^{(k)}$be the kth extended infinitesimal generator of the commutator$[X_1,X_2]$. Then

$$[X_1, X_2]^{(k)} = [X_1^{(k)}, X_2^{(k)}],$$

k = 1,2,… .

Consider the kth order ordinary differential equation (2.2.1), and let

$$x^* = \phi(x,y,a,b),$$
$$y^* = \psi(x,y,a,b),$$

be a two-parameter Lie group of transformations admitted by (2.2.1) with X_1, X_2as infinitesimal generators of the group. To reduce the order of the ordinary differential equation, we must use the appropriate generator first, since the use of incorrect order may not reduce the order of the ordinary differential equation by two. The commutator determines the appropriate generator to use first.

Suppose that $[X_1, X_2] = X_2$. So X_2 is the appropriate generator to use first.

Let $U(x,y)$, $V(x,y,y_1)$ be the invariants of $X_2^{(1)}$. by Theorem 2.2.1, $V_1 = \frac{dV}{dU}$ is a second order differential invariant of X_2 i.e. $X_2^{(2)}V_1 = 0$. Re-writing (2.2.1) in terms of the differential invariant, we obtain

$$Q(U,V,V_1,\ldots,V_{k-1}) = 0. \qquad (2.2.4)$$

The order of this equation is less than the order of (2.2.1) by one.

Now

$$[X_1, X_2]U = X_1X_2U – X_1X_2U =X_2U.$$

Since $X_1 \neq X_2$ and $X_2U = 0$, we get

$$X_1U = \alpha(U). \qquad (2.2.5)$$

From Theorem 2.2.5

$$[X_1, X_2]^{(1)} = [X_1^{(1)}, X_2^{(1)}],$$

so

$$[X_1^{(1)}, X_2^{(1)}]V = X_1^{(1)}X_2^{(1)}V – X_2^{(1)}X_1^{(1)}V = X_2^{(1)}V,$$

Since $X_1^{(1)} \neq X_2^{(1)}$and $X_2^{(1)}V = 0$, we have

$$X_1^{(1)}V = \beta(U,V). \qquad (2.2.6)$$

By the same way

$$[X_1^{(2)}, X_2^{(2)}]V_1 = X_1^{(2)}X_2^{(2)}V_1 - X_2^{(2)}X_1^{(2)}V_1 = X_2^{(2)}V_1,$$

then

$$X_1^{(2)}V_1 = \gamma(U,V,V_1). \qquad (2.2.7)$$

By using (2.2.5) – (2.2.7), we write $X_1^{(2)}$in terms of (U,V,V_1) coordinates, $X_1^{(2)}$ becomes

$$X_1^{(2)} = \alpha(U)\frac{\partial}{\partial U} + \beta(U,V)\frac{\partial}{\partial V} + \gamma(U,V,V_1)\frac{\partial}{\partial V_1}.$$

Let $u(U,V)$,$\nu(U,V,V_1)$ be invariants of $X_1^{(2)}$. The $\nu_1 = \frac{d\nu}{du}$is a third order differential invariant i.e. $X_1^{(3)}\nu_1 = 0$. By re-writing (2.2.4) in terms of u,ν and ν_1, we obtain

$$K(u, \nu,\nu_1,\ldots,\nu_{k-2}) = 0.$$

Hence we reduced the order of differential equation (2.2.1) by two. This method will be used in Chapter 3.

2.3 Invariant solutions

Consider an nth order ordinary differential equation

$$G\,(x,y,y_1,\ldots,y_n), \qquad (2.3.1)$$

which admits a one-parameter Lie group of transformations

$$x^* = \phi(x,y,a), \qquad (2.3.2a)$$

$$y^* = \psi(x,y,a), \qquad (2.3.2b)$$

with infinitesimal generator

$$X = \xi\,(x,y)\frac{\partial}{\partial x} + \eta\,(x,y)\frac{\partial}{\partial y}. \qquad (2.3.3)$$

Theorem 2.3.1. ([8], Theorem 2.2.7-1). A curve written in a solved form $g(x,y) = y - f(x) = 0$, is an invariant curve for (2.3.2a,b) if and only if

$$Xg(x,y) = \eta(x,y) - \xi(x,y)f'(x) = 0,$$

when $g(x,y) = y - f(x) = 0$, that is if and only if

$$\eta(x,f(x)) - \xi(x, f(x))\, f'(x) = 0. \quad (2.3.4)$$

Definition 2.3.2. ([10], Definition 1.9.3). A solution $y = \theta(x)$ is an invariant solution of the nth order ordinary differential equation

$$G(x,y,y_1,\ldots,y_n),$$

corresponding to infinitesimal generator (2.3.3) admitted by (2.3.1) if and only if

(i) $y = \theta(x)$ is an invariant curve of (2.3.3),

(ii) $y = \theta(x)$ solves (2.3.1).

The following is a result from theorem 2.3.1 and definition 2.3.2:

A solution $y = \theta(x)$ of (2.3.1) is an invariant solution if and only if it is a solution of the first order ordinary differential equation

$$y' = \frac{\eta(x,y)}{\xi(x,y)}. \quad (2.3.5)$$

By solving (2.3.5) we have the general solution in the form $J(x,y,c) = 0$, then substituting this solution in (2.3.1) the constant c is determined.

Bluman and Kumei [8]stated in the following theorem that it is not essential to solve (2.3.5) to obtain the invariant solutions for (2.3.1).

Theorem 2.3.3. ([8], Theorem 3.6-1). Suppose the ordinary differential equation (2.3.1) admits (2.3.3). Without loss of generality assume that $\xi \neq 0$. Let

$$\Upsilon = \frac{\partial}{\partial x} + \frac{\eta(x,y)}{\xi(x,y)}\frac{\partial}{\partial y},$$

and let

$$\Psi(x,y) = \frac{\eta(x,y)}{\xi(x,y)},$$

With $y_k = \Upsilon^{k-1}\Psi$, k = 1,2,…,n, consider the algebraic expression $Q(x,y)$ which is defined by

$$\begin{aligned} Q(x,y) &= y_n - G(x,y,y_1,\ldots,y_{n-1}) \\ &= \Upsilon^{n-1}\Psi - G(x,y,\Psi,\Upsilon\Psi,\ldots \Upsilon^{n-2}\Psi) \end{aligned}$$

Three cases arise for the algebraic equation $Q(x,y) = 0$:

(i) $Q(x,y) = 0$ defines no curves in the $x\,y$ -plane,

(ii) $Q(x,y) = 0$ is identically satisfied for all x,y,

(iii) $Q(x,y) = 0$ defines curves in the $x\,y$ -plane.

We describe the above three cases as follows :

1) In case (i), the differential equation (2.3.1) admits (2.3.3) has no invariant solution under (2.3.3).
2) In case (ii), any solution of (2.3.5) is an invariant solution of the differential equation (2.3.1) admits (2.3.3).
3) In case (iii), the invariant solution of the differential equation (2.3.1) admits (2.3.3) must satisfy $Q(x,y) = 0$ and conversely any curve satisfying $Q(x,y) = 0$ is an invariant solution of the differential equation (2.3.1) admits (2.3.3).

CHAPTER 3

APPLICATION OF SYMMETRY METHOD TO BOUNDARY LAYER EQUATIONS

3.0 Introduction

Some problems arising in engineering and industrial field are concerned with fluid dynamics. Engineers have a great interest to study the dynamical behavior of a fluid near a surface, which is known as boundary layer problems.

Two of the differential equations arising in boundary layer theory are the boundary layer equation (BLE) and Falkner-Skan equation (FSE).

In this Chapter, we will analyze boundary layer equations (BLE and FSE) using symmetry method. In Section 3.1 we obtain symmetry groups admitted by the boundary layer equations. In Section 3.2, we will reduce the order of boundary equations using the symmetry group and we will give a general solution of the boundary layer equation (BLE) for the special case $\beta = 2\alpha$. Constructing invariant solutions to boundary layer equation (BLE) is given in Section 3.3.

3.1 Symmetry Group of Boundary Layer Equations

Here we use the method outlined in Section 2.1 to obtain symmetry groups of boundary layer equations (BLE and FSE).

3.1.1 Symmetry group of BLE

Assume that the boundary layer equation (BLE)

$$f''' + \beta f f'' - \alpha f'^2 = 0, \tag{3.1.1}$$

where f is a reduced stream function, z is a similarity variable, α and β are constants, and prime denotes a derivative with respect to z. The BLE admits a one-parameter Lie group of transformation with infinitesimal generator

$$X = \xi(z,f)\frac{\partial}{\partial z} + \eta(z,f)\frac{\partial}{\partial f}. \qquad (3.1.2)$$

From the infinitesimal Criterion for Invariance, the BLE admits a one-parameter Lie group of transformation with infinitesimal generator (3.1.2) if and only if

$$X^{(3)}(f_3 + \beta f f_2 - \alpha f_1^2) = 0,$$

when $f_3 + \beta f f_2 - \alpha f_1^2 = 0$, where $f_k(z) = \frac{\partial^k f}{\partial z^k}$, k = 1,2,3.

Thus, we obtain the following equation

$$\begin{aligned} &\eta_{zzz} + (3\eta_{zzf} - \xi_{zzz}) f_1 + 3(\eta_{zff} - \xi_{zzf}) f_1^2 + (\eta_{fff} - 3\xi_{zff}) f_1^3 - \\ &\xi_{fff} f_1^4 + 3(\eta_{zf} - \xi_{zz}) f_2 + 3(\eta_{ff} - 3\xi_{zf}) f_1 f_2 - 6\xi_{ff} f_1^2 f_2 \\ &- 3\xi_f f_2^2 - \beta(\eta_f - 3\xi_z) f f_2 + \alpha(\eta_f - 3\xi_z) f_1^2 + 4\beta\xi_f f f_1 f_2 \\ &- 4\alpha\xi_f f_1^3 + \beta\eta_{zz} f + \beta(2\eta_{zf} - \xi_{zz}) f f_1 + \beta(\eta_{ff} - 2\xi_{zf}) f f_1^2 \\ &- \beta\xi_{ff} f f_1^3 + \beta(\eta_f - 2\xi_z) f f_2 - 3\beta\xi_f f f_1 f_2 - 2\alpha\eta_z f_1 - 2\alpha(\eta_f - \\ &\xi_z) f_1^2 + 2\alpha\xi_f f_1^3 + \beta\eta f_2 = 0. \end{aligned} \qquad (3.1.3)$$

Here we have substituted for each f_3 in the previous equation by $(-\beta f f_2 + \alpha f_1^2)$. Equating the coefficients of various powers of f_1 and f_2, we obtain the following equations

$$\eta_{zzz} + \beta\eta_{zz} f = 0,$$

$$3\eta_{zzf} - \xi_{zzz} + \beta(2\eta_{zf} - \xi_{zz}) f - 2\alpha\eta_z = 0,$$

$$3(\eta_{zff} - \xi_{zzf}) + \alpha(\eta_f - 3\xi_z) + \beta(\eta_{ff} - 2\xi_{zf}) f - 2\alpha(\eta_f - \xi_z) = 0,$$

$$\eta_{fff} - 3\xi_{zff} - 4\alpha\xi_f - \beta\xi_{ff} f + 2\alpha\xi_f = 0,$$

$$3(\eta_{zf} - \xi_{zz}) - \beta(\eta_f - 3\xi_z) f + \beta(\eta_f - 2\xi_z) f + \beta\eta = 0,$$

$$3(\eta_{ff} - 3\xi_{zf}) + \beta\xi_f f = 0,$$

$$\xi_{fff} = \xi_{ff} = \xi_f = 0.$$

Solving the above equations for $\xi(z,f)$ and $\eta(z,f)$ we find

$$\xi(z,f) = -c_2 z + c_1, \qquad (3.1.4)$$

$$\eta(z,f) = c_2 f, \qquad (3.1.5)$$

where c_1 and c_2 are arbitrary constants. Then the infinitesimal generator of equation (3.1.1) becomes

$$X = (-c_2 z + c_1)\frac{\partial}{\partial z} + c_2 f\frac{\partial}{\partial f}$$

$$= c_1 X_1 + c_3 X_2,$$

where $c_3 = -c_2$, $X_1 = \frac{\partial}{\partial z}$ and $X_2 = z\frac{\partial}{\partial z} - f\frac{\partial}{\partial f}$. X_1 and X_2 are two infinitesimal generators from the infinitesimal generator X, which are also admitted by equation (3.1.1). Hence the symmetry group of equation (3.1.1) is the two parameter Lie group of transformation of the form:

$$z^* = e^b z + a, \quad (3.1.6a)$$

$$f^* = e^{-b} f. \quad (3.1.6b)$$

3.1.2 Symmetry group of FSE

Assume that the Falkner-Skan equation (FSE)

$$f''' + ff'' + 2(1 - f'^2) = 0, \quad (3.1.7)$$

where f is a reduced stream function, z is a similarity variable and prime denotes a derivative with respect to z. The FSE admits a one-parameter Lie group of transformation with infinitesimal generator (3.1.2) if and only if

$$X^{(3)}(f_3 + ff_2 + 2(1 - f_1^2)) = 0,$$

when $f_3 + ff_2 + 2(1 - f_1^2) = 0$, where $f_k(z) = \frac{\partial^k f}{\partial z^k}$, k = 1,2,3.

Thus, we obtain the following equation

$$\begin{aligned}
&\eta_{zzz} + (3\eta_{zzf} - \xi_{zzz})\, f_1 + 3(\eta_{zff} - \xi_{zzf})f_1^2 + (\eta_{fff} - 3\xi_{zff})f_1^3 - \xi_{fff}f_1^4 \\
&+ 3(\eta_{zf} - \xi_{zz})f_2 + 3(\eta_{ff} - 3\xi_{zf})f_1 f_2 - 6\xi_{ff}f_1^2 f_2 \\
&- 3\xi_f f_2^2 - 2(\eta_f - 3\xi_z) - (\eta_f - 3\,\xi_z)\, f f_2 + 2(\eta_f - 3\xi_z)f_1^2 + \\
&\xi_f f f_1 f_2 + 8\xi_f f_1 - 4\xi_f f_1^3 + \eta_{zz} f + (2\eta_{zf} - \xi_{zz}) f f_1 + (\eta_{ff} - \\
&2\xi_{zf})\, f f_1^2 - \xi_{ff} f f_1^3 + (\eta_f - 2\,\xi_z)\, f f_2 - 4\eta_z f_1 - 4\,(\eta_f - \xi_z)f_1^2 + \\
&\eta f_2 = 0.
\end{aligned} \quad (3.1.8)$$

Here we have substituted for each f_3 in the previous equation by $(-ff_2 - 2(1 - f_1'^2)$. Equating the coefficients of various powers of f_1 and f_2, we obtain the following equations

$$
\begin{gathered}
\eta_{zzz} - 2(\eta_f - 3\xi_z) + \eta_{zz} f = 0, \\
3\eta_{zzf} - \xi_{zzz} - 4\eta_z + (2\eta_{zf} - \xi_{zz})f + 8\xi_f = 0, \\
3(\eta_{zff} - \xi_{zzf}) + 2(\eta_f - 3\xi_z) + (\eta_{ff} - 2\xi_{zf})f - 4(\eta_f - \xi_z) = 0, \\
\eta_{fff} - 3\xi_{zff} - \xi_{ff} f - 4\xi_f = 0, \\
3(\eta_{zf} - \xi_{zz}) - (\eta_f - 3\xi_z) f + (\eta_f - 2\xi_z) f + \eta = 0, \\
3(\eta_{ff} - 3\xi_{zf}) + \xi_f f = 0, \\
\xi_{fff} = \xi_{ff} = \xi_f = 0.
\end{gathered}
$$

Solving the above equations for $\xi(z, f)$ and $\eta(z, f)$ we find

$$\xi(z, f) = c, \tag{3.1.9}$$

$$\eta(z, f) = 0, \tag{3.1.10}$$

where c is an arbitrary constant. Then the infinitesimal generator of equation (3.1.7) becomes

$$X = \frac{\partial}{\partial z}.$$

Hence the symmetry group of equation (3.1.7) is the one-parameter Lie group of transformation of the form:

$$z^* = z + a, \tag{3.1.11a}$$

$$f^* = f. \tag{3.1.11b}$$

3.2 Reduction In Order of Boundary Layer Equations

In this section, we will use symmetry groups of boundary layer equations (BLE and FSE) to reduce their order.

3.2.1 Reduction in order of BLE

The BLE (3.1.1) admits a two parameter Lie group of transformations. Therefore the order of the differential equation may be reduced by two. To determine which infinitesimal generator to use first, we compute the commutator:

$$[X_1, X_2] = X_1X_2 - X_2X_1 = X_1.$$

We begin with the generator X_1. The first extended infinitesimal generator of X_1 is

$$X_1^{(1)} = \frac{\partial}{\partial z}. \tag{3.2.1}$$

The invariants of (3.2.1) are

$$q\,(z,f) = f,$$

$$p(z,f,f_1) = f_1.$$

Then

$$p_1 = \frac{f_2}{f_1},$$

is the second order differential invariant satisfying $X_1^{(2)}p_1 = 0$.

Now

$$f_1 = p,$$

$$f_2 = p_1p\,,$$

$$f_3 = p_2p^2 + p_1^2p.$$

By substituting the above into equation (3.1.2), we obtain the second order ODE:

$$pp_2 + p_1^2 + \beta qp_1 - \alpha p = 0. \tag{3.2.2}$$

The infinitesimal generator X_2 is admitted by equation (3.2.2) since it is admitted by equation (3.1.1). The second extended infinitesimal generator of X_2 in terms of (q,p,p_1) is

$$X_2^{(2)} = -q\frac{\partial}{\partial q} - 2p\frac{\partial}{\partial p} - p_1\frac{\partial}{\partial p_1}.$$

Let $w(q,p)$ and $g(q,p,p_1)$ be the invariants of X_2 such that

$$X_2^{(1)}w(q,p) = 0,$$

and

$$X_2^{(2)}g(q,p,p_1) = 0.$$

Therefore

$$w(q,p) = q^{-2}p,$$

and

$$g(q,p,p_1) = q^{-1}p_1.$$

Now

$$p_1 = qg,$$

$$p_2 = g + g_1(-2w + g).$$

Rewriting equation (3.2.2) in terms of (w,g,g_1), we obtain the first order ODE:

$$g_1 = \frac{g^2 + \beta g + wg - \alpha w}{-2w^2 + wg}. \tag{3.2.3}$$

If $g = \mathrm{N}(w,c_1)$ is the general solution of equation (3.2.3), then the first order ODE

$$g = q^{-1}p_1 = \mathrm{N}(q^{-2}p, c_4), \tag{3.2.4}$$

may be solved by the use of canonical coordinates. Equation (3.2.3) admits the infinitesimal generator $\mathrm{X}_2^{(1)}$, then the canonical coordinates $r(q,p)$ and $s(q,p)$ satisfy the following

$$\mathrm{X}_2^{(1)} r(q,p) = 0, \tag{3.2.5}$$

and

$$\mathrm{X}_2^{(1)} s(q,p) = 1. \tag{3.2.6}$$

By solving (3.2.5) and (3.2.6), we obtain

$$r(q,p) = q^{-2}p,$$

and

$$s(q,p) = -\frac{1}{2}\ln p,$$

then

$$\frac{ds}{dr} = -\frac{N(r,c_4)}{2(-2r^2 + rN(r,c_4))}.$$

This leads to the first quadrature

$$p = c_5 \exp\left[\int^{q^{-2}p} \frac{N(\rho,c_4)}{(\rho N(\rho,c_4) - 2\rho^2)}\, d\rho\right]. \tag{3.2.7}$$

Expressing equation (3.2.7) in terms of (z,f,f_1), we have

$$f_1 = c_5 \exp\left[\int^{f^{-2}f_1} \frac{N(\rho,c_4)}{(\rho N(\rho,c_4)-2\rho^2)}\, d\rho\right]. \quad (3.2.8)$$

Writing (3.2.8) in a solved form

$$f_1 = I(f, c_4, c_5), \quad (3.2.9)$$

where c_4 and c_5 are arbitrary constants, then the solution (second quadrature) of the first order ODE (3.2.9) is

$$\int \frac{df}{I(f,c_4,c_5)} = z + c_6.$$

Special Case $\boldsymbol{\beta = 2\alpha}$

We find a general solution of equation (3.1.1) for the special case when $\beta = 2\alpha$. Equation (3.2.3) becomes

$$(g - 2w)wg_1 + g\,(w + 2\,\alpha) + g^2 - \alpha w = 0, \quad (3.2.10)$$

This equation has a special solution

$$g(w) = \frac{w}{2}\,, \quad (3.2.11)$$

but

$$g = q^{-1}p_1,$$

and

$$w = q^{-2}p,$$

then

$$p_1 = \frac{1}{2}q^{-1}p. \quad (3.2.12)$$

By solving equation (3.2.12) for p, we obtain

$$p = c_7 q^{\frac{1}{2}}\,,$$

then

$$f_1 = c_7 f^{\frac{1}{2}}\,. \quad (3.2.13)$$

Now, we solve equation (3.2.13) for f. The solution of the equation (3.2.13), that is the general solution of the BLE is

$$f = (c_7\frac{z}{2} + c_8)^2. \quad (3.2.14)$$

Thus, we have reduced the order of the BLE by two with two quadratures and obtained a general solution when $\beta = 2\alpha$.

3.2.2 Reduction in order of FSE

The FSE (3.1.7) admits a one-parameter Lie group of transformations. Therefore the order of the FSE may be reduced by one. The first extended infinitesimal generator of X admitted by FSEis

$$X^{(1)} = \frac{\partial}{\partial z}. \qquad (3.2.15)$$

The invariants of (3.2.15) are

$$q\,(z,f) = f,$$
$$p(z,f,f_1) = f_1.$$

Then

$$p_1 = \frac{f_2}{f_1},$$

is the second order differential invariant satisfying $X^{(2)}p_1 = 0$.

Now

$$f_1 = p,$$
$$f_2 = p_1 p\,,$$
$$f_3 = p_2 p^2 + p_1^2 p.$$

By substituting the above into equation (3.1.7), we obtain the second order ODE:

$$p^2(p_2 - 2) + (p_1 + q)pp_1 + 2 = 0.$$

Here the symmetry method gives no important results for the FSE.

3.3 Invariant Solutions to Boundary Layer Equations

In this section, we find the invariant solutions of the boundary layer equations. We begin with the FSE (3.1.7)

$$f''' + ff'' + 2(1 - f'^2) = 0,$$

by using theorem 2.3.1 and definition 2.3.2 or by using theorem 2.3.3 the result is an invariant solution

$$f = c,$$

which is a trivial solution to Falkner-Skan equation.

Next, we find the invariant solutions of the BLE (3.1.1)

$$f''' + \beta f f'' - \alpha f'^2 = 0,$$

that admits a two-parameter Lie group of transformation with the infinitesimal generator

$$X = (-c_2 z + c_1)\frac{\partial}{\partial z} + c_2 f\frac{\partial}{\partial f}. \qquad (3.3.1)$$

There are three cases for the constants c_1 and c_2 in (3.3.1):

i)For $c_2 = 0$, we use theorem 2.3.1 and definition 2.3.2 to obtain the invariant solutions, we have

$$f = c,$$

is an invariant solution of (3.1.1).

ii) For $c_1 = 0$, also by using theorem 2.3.1 and definition 2.3.2, we obtain

$$f = \frac{6}{(2\beta - \alpha)z},$$

is an invariant solution of (3.1.1).

iii) For $c_1 \neq 0$, and $c_2 \neq 0$, we use theorem 2.3.3

$$\gamma = \frac{\partial}{\partial z} + \frac{c_2}{-c_2 z + c_1} f\frac{\partial}{\partial f}.$$

Let $\lambda = \frac{c_1}{c_2} > 0$. Then

$$\gamma = \frac{\partial}{\partial z} + \frac{f}{-z+\lambda}\frac{\partial}{\partial f},$$

and

$$\Psi(z, f) = \frac{f}{-z+\lambda}.$$

$$\gamma\Psi = \frac{f^2}{(-z+\lambda)^2}.$$

$$\gamma^2\Psi = \frac{4f}{(-z+\lambda)^3}.$$

Then

$$Q(z,f) = \gamma^2\Psi + \beta f\gamma\Psi - \alpha\Psi^2 = 0.$$

By solving the above equation, we have

$$f = \frac{-6}{(2\beta - \alpha)(-z+\lambda)}, \tag{3.3.2}$$

is an invariant solution of (3.1.1).

The invariant solution (3.3.2) satisfies the boundary layer equation (3.1.1) under the following boundary conditions:

$$f(0) = \frac{-6}{(2\beta - \alpha)\lambda} = f_0, \quad f'(0) = \frac{-6}{(2\beta - \alpha)\lambda^2} = \omega, \text{ and } f'(\infty) = 0,$$

wheref_0 is the wall mass transfer parameter showing the strength of the mass transfer at the wall (surface), and ω the wall movement parameter indicating the strength of the wall stretching. The invariant solution satisfies equation (3.1.1) and the above boundary conditions only in the case where$2\beta > \alpha$. Thus for $f_0 < 0$ which gives the boundary condition for mass injection and $\omega < 0$ that corresponds to moving surface in opposite direction to the mainstream. Note that the invariant solution does not satisfy the equation for $2\beta \leq \alpha$. Moreover does not satisfy the boundary conditions for $\omega = 0$ the case of fixed surface, $\omega > 0$ the case for moving surface in same direction to the mainstream, $f_0 = 0$ the case of impermeable surface, and $f_0 > 0$ the case for mass suction.

Conclusion

The main aims of this research has been to study a new method to solve an ordinary differential equations and analyze the boundary layer equations

$$f''' + \beta f f'' - \alpha f'^2 = 0,$$

$$f''' + f f'' + 2(1 - f'^2) = 0,$$

using this method which is a group theoretical method known as symmetry method. We have outlined the basic concepts and definitions of the method. We have applied the method to obtain the symmetry groups admitted by the boundary layer equations. Under the symmetry groups we have reduced the order of the boundary layer equation to quadrature, also we have reduced the order of the Falkner-Skan equation to the second order. Furthermore, we have given a general solution of the boundary layer equation for the special case when $\beta = 2\alpha$. Finally, we have constructed invariant solutions of the boundary layer equation from the symmetry group, which satisfies equation (3.1.2) only if $2\beta > \alpha$. This invariant solution is different than the solution obtained in [5] known as Cranes' solution. In addition, we have discussed the boundary conditions that the invariant solution of equation (3.1.2) satisfies.

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